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Solution to the initial value problem of the ultradiscrete periodic Toda equation

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Abstract
We present an expression for the solution to the initial value problem for the ultradiscrete periodic Toda equation. The expression provides explicit forms of all dependent variables of the equation, while the previously known solutions give only half of the dependent variables while the others have to be determined implicitly using the conserved quantities.

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1. Introduction

The ultradiscrete periodic Toda equation (udp Toda equation) is obtained from the discrete Toda equation [1]—which is a well-known integrable partial difference equation—by imposing a periodic boundary condition and applying a limiting procedure called ultradiscretization [2, 3]. Through ultradiscretization we can construct piecewise linear equations or cellular automata from continuous equations. The udp Toda equation describes a time evolution of a periodic box–ball system (PBBS), which is a dynamical system of balls in a one-dimensional array of boxes with a periodic boundary condition [4, 5].

Recently, using the soliton solution of the ultradiscrete KdV equation, we obtained the solution to the initial value problem (IVP) for the PBBS and ultradiscrete Toda molecule equation [6]. In this paper, using the ideas in [6], we derive an expression for the solution to the IVP for the udp Toda equation. We note that the same problem was also studied in [4] and [5]; the authors considered the inverse ultradiscretization and solved the initial value problem of the discrete Toda equation, using the linearization of the dynamics on a Jacobian variety associated with a hyperelliptic curve determined by the corresponding spectral problem. In that method, however, only half of the dependent variables are determined in the form of ultradiscrete theta functions and the others have to be determined by solving rather difficult algebraic equations determined by the conserved quantities, which is almost impossible in practice. In comparison, our method will give all dependent variables in explicit forms.

2. Statement of the result

Let N be a positive integer. The periodic discrete Toda equation is a system of equations

$$I_m^{t+1} = I_m^t + V_m^t - V_{m+1}^{t+1}, \qquad V_m^{t+1} = \frac{I_{m-1}^t V_m^t}{I_m^{t+1}},$$

where $m = 1, 2, \ldots, N$, and $t \in \mathbb{Z}$, with the boundary condition

$$I_{m+N}^t = I_m^t, \qquad V_{m+N}^t = V_m^t.$$

The variables I_m^t and V_m^t are real valued. Suppose that the system has a one-parameter family of real positive solutions $\left\{I_m^t(\epsilon), V_m^t(\epsilon)\right\}_{\epsilon>0}$ that satisfies

$$0 \leqslant \lim_{\epsilon\to+0} \prod_{m=1}^{N} \frac{V_m^t(\epsilon)}{I_m^t(\epsilon)} < 1,$$

and moreover that the limits

$$Q_m^t := \lim_{\epsilon\to+0} -\epsilon \log I_m^t(\epsilon), \qquad E_m^t := \lim_{\epsilon\to+0} -\epsilon \log V_m^t(\epsilon)$$

exist. Then the new variables satisfy the following set of equations [4, 5], called the udp Toda equation:

$$Q_m^{t+1} = \min\left\{E_m^t, X_m^t + Q_m^t\right\}, \tag{1}$$

$$E_m^{t+1} = Q_{m-1}^t + E_m^t - Q_m^{t+1}, \tag{2}$$

$$X_m^t = \max\left\{0, \max_{1\leqslant k\leqslant N-1}\left\{\sum_{i=1}^{k}\left(Q_{m+i}^t - E_{m+i}^t\right)\right\}\right\} \tag{3}$$

and the inequality

$$\sum_{m=1}^{N} Q_m^t < \sum_{m=1}^{N} E_m^t. \tag{4}$$

The index m in the Q_m^t and E_m^t is to be read modulo N. A marked property of this system is that it preserves positivity and integrality of the variables Q_m^t and E_m^t. We are interested in the case where all the variables are positive integers.

An initial state of the system is given by a $2N$-tuple of positive integers, $(q_1, e_1, \ldots, q_N, e_N)$, where $Q_m^0 = q_m, E_m^0 = e_m$ $(m = 1, \ldots, N)$. We shall picture such a $2N$-tuple as a lattice path from (0, 0) to $(L, L - 2M)$, where

$$L = \sum_{m=1}^{N} e_m + \sum_{m=1}^{N} q_m, \qquad M = \sum_{m=1}^{N} q_m$$

(therefore $L - 2M = \sum e_m - \sum q_m$); each q_m represents a displacement of q_m units to the right and q_m units down (i.e., a 'downhill' segment), and each e_m represents a displacement of e_m units to the right and e_m units up (i.e., an 'uphill' segment). We extend this procedure continuously outside the interval $[0, L]$. For example,

$$(q_1, e_1, q_2, e_2, q_3, e_3, q_4, e_4) = (1, 4, 3, 2, 4, 6, 4, 7)$$

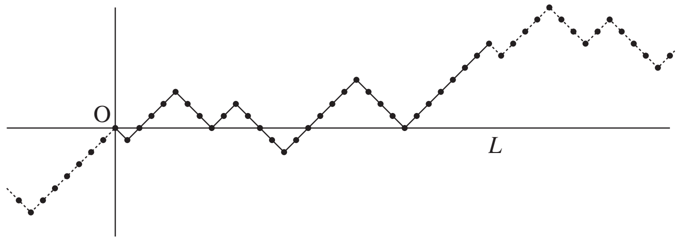

Figure 1. A lattice path which represents $(q_1, e_1, q_2, e_2, q_3, e_3, q_4, e_4) = (1, 4, 3, 2, 4, 6, 4, 7)$, where $L = 31$.

is graphed as in figure 1. For a given $2N$-tuple and for each $m = 1, \ldots, N$, we write

$$v_m = \sum_{1\leqslant i\leqslant m} q_i + \sum_{1\leqslant i\leqslant m-1} e_i = (\text{the } x\text{-coordinate of the `}m\text{th minimum'}), \tag{5}$$

$$l_m = \max\left\{\begin{pmatrix}\text{the } x\text{-coordinate of the position where the}\\ \text{height, relative to that at } v_m, \text{ becomes } -1\\ \text{for the first time when we go along the path}\\ \text{leftward starting from } v_m\end{pmatrix}, v_m - L\right\},$$

$$r_m = \min\left\{\begin{pmatrix}\text{the } x\text{-coordinate of the position where the}\\ \text{height, relative to that at } v_m, \text{ becomes } 0\\ \text{for the first time when we go along the path}\\ \text{rightward starting from } v_m\end{pmatrix}, v_m + L\right\},$$

and

$$W_m^L = \begin{pmatrix}\text{the maximum height, relative to that at } v_m,\\ \text{in the lattice path on the interval } [l_m, v_m]\end{pmatrix}, \tag{6}$$

$$W_m^R = \begin{pmatrix}\text{the maximum height, relative to that at } v_m,\\ \text{in the lattice path on the interval } [v_m, r_m]\end{pmatrix}, \tag{7}$$

and we define

$$W_m = \min\left\{W_m^L, W_m^R\right\}. \tag{8}$$

In our example above, $L = 31$ and $v_1 = 1, v_2 = 8, v_3 = 14, v_4 = 24$; $l_1 = -2, l_2 = 1, l_3 = -3, l_4 = 15$; $r_1 = 13, r_2 = 12, r_3 = 45(= v_3 + L), r_4 = 55(= v_4 + L)$; hence, $W_1^L = 1$, $W_2^L = 3$, $W_3^L = 5$, $W_4^L = 4$; $W_1^R = 4$, $W_2^R = 2$, $W_3^R = 12$, $W_4^R = 11$; $W_1 = 1, W_2 = 2$, $W_3 = 5$, $W_4 = 4$. The following is the main statement, the proof of which is given in the next section.

Theorem 2.1. *Let N be a positive integer. Let $q_1, e_1, \ldots, q_N, e_N$ be positive integers that satisfy*

$$\sum_{1\leqslant m\leqslant N} q_m < \sum_{1\leqslant m\leqslant N} e_m. \tag{9}$$

We assume further that

$$\sum_{i\leqslant m\leqslant N} q_m < \sum_{i\leqslant m\leqslant N} e_m \qquad (i = 2, 3, \ldots, N); \tag{10}$$

cf the remark below. The unique solution Q^t_m, E^t_m $(m = 1, 2, \ldots, N)$*, of the initial value problem (1)–(3) with*

$$Q^0_m = q_m, \qquad E^0_m = e_m \qquad (m = 1, 2, \ldots, N)$$

is given by

$$Q^t_m = \Psi^t_{m-1} - \Psi^t_m - \Psi^{t-1}_{m-1} + \Psi^{t-1}_m, \qquad E^t_m = \Psi^{t-1}_m - \Psi^{t-1}_{m+1} - \Psi^t_{m-1} + \Psi^t_m,$$

where

$$\Psi^t_m = \max_{\substack{(n_1,\ldots,n_N)\in\mathbb{Z}^N \\ \sum n_i = m}} \left[\sum_{i=1}^N n_i(b_i - tW_i) - \sum_{i=1}^N \sum_{j=1}^N n_i \Xi_{ij} n_j \right], \tag{11}$$

$$b_i = -a_i - 2 \sum_{i+1\leqslant j\leqslant N} \min\{W_i, W_j\} - W_i + \sum_{1\leqslant j\leqslant N} \min\{W_i, W_j\}, \tag{12}$$

$$\Xi_{ij} = \left(\frac{L}{2} - \sum_{1\leqslant k\leqslant N} \min\{W_i, W_k\} \right) \delta_{ij} + \min\{W_i, W_j\}, \tag{13}$$

$$a_i = \sum_{1\leqslant j\leqslant i} q_j + \sum_{1\leqslant j\leqslant i-1} e_j + 1 \qquad (= v_i + 1) \tag{14}$$

$(i, j = 1, 2, \ldots, N)$*, the* W_i*'s are those for the* $2N$*-tuple* $(q_1, e_1, \ldots, q_N, e_N)$*, and* δ_{ij} *is Kronecker's delta.*

Note that, in the equations above, changing the values of a_i for all i simultaneously by the same amount, $a_i \to a_i + \alpha$, causes changes in b_i and Ψ^t_m as in

$$b_i \to b_i - \alpha, \qquad \Psi^t_m \to \Psi^t_m - m\alpha;$$

but the values of Q^t_m and E^t_m remain unchanged.

Remark 2.2. The assumption (10) is not an essential restriction. If a given set of positive integers satisfies (9) but does not satisfy (10), we calculate

$$\mu := \max_{2\leqslant i\leqslant N} \sum_{i\leqslant m\leqslant N} (q_m - e_m)$$

($\mu \geqslant 0$ by assumption) and

$$i_0 := \min \left\{ i \,\middle|\, 2 \leqslant i \leqslant N, \sum_{i\leqslant m\leqslant N} (q_m - e_m) = \mu \right\}.$$

If we then shift the indices cyclically such that $q'_N = q_{i_0-1}, e'_N = e_{i_0-1}, q'_{N-1} = q_{i_0-2}, e'_{N-1} = e_{i_0-2}, \ldots, q'_2 = q_{i_0+1}, e'_2 = e_{i_0+1}, q'_1 = q_{i_0}, e'_1 = e_{i_0}$, the new set of integers, $q'_1, e'_1, \ldots, q'_N, e'_N$ satisfies both (9) and (10).

The W_m, defined in (8), can be computed directly, without the use of any pictures.

Proposition 2.3. *Let* N *be a positive integer, and let* $(q_1, e_1, \ldots, q_N, e_N)$ *be a* $2N$*-tuple of positive integers.*

(1) Write

$$\mathrm{V}_m^{(k)L} = \min_{1\leqslant j\leqslant k}\left\{\sum_{1\leqslant i\leqslant j}(q_{m-i+1}-e_{m-i})\right\}$$

for $k = 1, \ldots, N-1$, *and*

$$\Lambda_m^{(1)L} = q_m,$$

$$\Lambda_m^{(k)L} = \begin{cases} \displaystyle\sum_{m-k+1\leqslant i\leqslant m-1}(q_i - e_i) + q_m & \text{if} \quad \mathrm{V}_m^{(k-1)L} \geqslant 0, \\ -1 & \text{if} \quad \mathrm{V}_m^{(k-1)L} \leqslant -1 \end{cases}$$

for $k = 2, \ldots, N$. *Then,* W_m^L *(6) is given by*

$$W_m^L = \max\left\{\Lambda_m^{(k)L} \middle| 1 \leqslant k \leqslant N\right\}.$$

(2) Write

$$\mathrm{V}_m^{(k)R} = \min_{1\leqslant j\leqslant k}\left\{\sum_{1\leqslant i\leqslant j}(e_{m+i-1}-q_{m+i})\right\}$$

for $k = 1, \ldots, N-1$, *and*

$$\Lambda_m^{(1)R} = e_m,$$

$$\Lambda_m^{(k)R} = \begin{cases} e_m + \displaystyle\sum_{m+1\leqslant i\leqslant m+k-1}(e_i - q_i) & \text{if} \quad \mathrm{V}_m^{(k-1)R} \geqslant 1, \\ -1 & \text{if} \quad \mathrm{V}_m^{(k-1)R} \leqslant 0 \end{cases}$$

for $k = 2, \ldots, N$. *Then,* W_m^R *(7) is given by*

$$W_m^R = \max\left\{\Lambda_m^{(k)R} \middle| 1 \leqslant k \leqslant N\right\}.$$

(Note that the value -1 *assigned to* $\Lambda_m^{(k)L,R}$ *in some cases is artificial; one can assign any number not greater than* q_m *(resp.* e_m*) to the same effect.)*

Proof. In the lattice path, for each $m = 1, \ldots, N$, we call the minimum at v_m (5) the mth minimum, and the maximum at $\lambda_m := v_m + e_m$ the mth peak; and, for each $s \in \mathbb{Z}$, we call the minimum at $v_m + sL$ the $(m+sN)$th minimum, and the maximum at $\lambda_m + sL$ the $(m+sN)$th peak. Let h denote the 'height function' of the path (i.e., the coordinate of a point on the path at x is $(x, h(x))$; see figure 2). Then, the first part of the claim is obvious if we note the following:

$$\sum_{1\leqslant i\leqslant j}(q_{m-i+1}-e_{m-i}) = h(v_{m-j}) - h(v_m),$$

$$\begin{aligned}\mathrm{V}_m^{(k)L} &= \min_{1\leqslant j\leqslant k} h(v_{m-j}) - h(v_m) \\ &= \begin{pmatrix}\text{height of the lowest minimum among the } (m-1)\text{th to} \\ (m-k)\text{th minima, relative to that of the } m\text{th one}\end{pmatrix},\end{aligned}$$

and

$$\sum_{m-k+1\leqslant i\leqslant m-1}(q_i - e_i) + q_m = h(\lambda_{m-k}) - h(v_m).$$

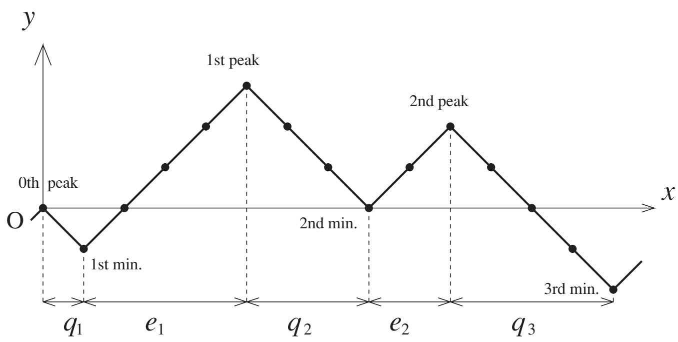

Figure 2. Peaks and minima of a lattice path $(q_1, e_1, q_2, e_2, \ldots)$; the path is identified with the graph of a function h (this is the definition of h).

The second part of the claim follows, since

$$\sum_{1\leqslant i\leqslant j} (e_{m+i-1} - q_{m+i}) = h(v_{m+j}) - h(v_m),$$

$$\begin{aligned} \mathrm{V}_m^{(k)R} &= \min_{1\leqslant j\leqslant k} h(v_{m+j}) - h(v_m) \\ &= \begin{pmatrix} \text{height of the lowest minimum among the } (m+1)\text{th to} \\ (m+k)\text{th minima, relative to that of the } m\text{th one} \end{pmatrix}, \end{aligned}$$

and

$$e_m + \sum_{m+1\leqslant i\leqslant m+k-1} (e_i - q_i) = h(\lambda_{m+k-1}) - h(v_m). \qquad \square$$

3. Proof of the theorem

3.1. Outline

We assume familiarity with the PBBS and the (infinite) box–ball system, as described, for example, in [4–6].

Let $q_1, e_1, \ldots, q_N, e_N$ be positive integers that satisfy $\sum q_m < \sum e_m$. It is known that the udp Toda equations (1)–(3) with the initial condition

$$Q_m^0 = q_m, \qquad E_m^0 = e_m \qquad (m = 1, \ldots, N)$$

describe the time evolution of the PBBS whose initial state is given by a finite sequence

$$\underbrace{1, \ldots, 1}_{q_1}, \underbrace{0, \ldots, 0}_{e_1}, \underbrace{1, \ldots, 1}_{q_2}, \underbrace{0, \ldots, 0}_{e_2}, \ldots, \underbrace{1, \ldots, 1}_{q_N}, \underbrace{0, \ldots, 0}_{e_N} \qquad (15)$$

where each ‘1’ stands for a filled box, each ‘0’ for empty one, and $L = \sum e_m + \sum q_m$ corresponds to the system length (or the number of boxes). At time t, the size of each block of consecutive 1’s is identified with Q_m^t and that of each block of consecutive 0’s with E_m^t [4, 5]. Such a state of the PBBS can be considered as an infinite sequence of 0’s and 1’s, or a

$$g \;=\; 000110001111000111000$$
$$T_L\, g \;=\; 100001100000111000111$$

Figure 3. The time evolution operator T_L of the PBBS.

$$g \;=\; \ldots 000110001111000111000000 \ldots$$
$$T\, g \;=\; \ldots 000001100000111000111100 \ldots$$

Figure 4. The time evolution operator T of the BBS.

mapping of $\mathbb{Z}$ into $\{0, 1\}$, periodic of period L; the state (15) is then identified with a sequence f that satisfies the following:

$$f \text{ is 1 on } \bigcup_{m=1}^{N} \left[\sum_{1\leqslant k\leqslant m-1} (q_k + e_k) + 1,\ \sum_{1\leqslant k\leqslant m-1} (q_k + e_k) + q_m \right],$$

$$f \text{ is 0 on } \bigcup_{m=1}^{N} \left[\sum_{1\leqslant k\leqslant m-1} (q_k + e_k) + q_m + 1,\ \sum_{1\leqslant k\leqslant m-1} (q_k + e_k) + q_m + e_m \right]$$

(where $[a, b]$ denotes the set of integers n that satisfy $a \leqslant n \leqslant b$) and $f(j + L) = f(j)$ for $j \in \mathbb{Z}$. The time evolution operator will be denoted by T_L (see figure 3).

Now we introduce a positive integer S; we shall let it tend to $+\infty$ later. Let f_S be a sequence defined by

$$f_S(j) = \begin{cases} f(j) & \text{if} \quad -SL + 1 \leqslant j \leqslant (S + 1)L, \\ 0 & \text{otherwise;} \end{cases}$$

in other words, f_S consists of $2S+1$ copies of $f|_{[1,L]}$ where the interval $[1, L]$ is in the middle, and zeros outside. We now think of f_S as a state of the (ordinary, or infinite) box–ball system (BBS), and consider its time evolution, $T^t f_S = T(T(\cdots T(f_S)\cdots))$, $t = 0, 1, 2, \ldots$ (For the definition of T see figure 4; or section 3 in [6]). Then, one should observe that the blocks of 1's and 0's of $T^t f_S$ on and near the interval $[1, L]$ will behave in the same manner as those of $T_L^t f$ until the effect of the left boundary at the initial time $t = 0$ reaches there; hence, up to this point in time, states of the PBBS, therefore the variables of the udp Toda, Q_m^t and E_m^t, will be described by those of the BBS.

To be more precise, there are $(2S+1)N$ blocks of consecutive 1's in $T^t f_S$; let $\tilde{a}_i(t)$ denote the position of the ith block ($i = 1, 2, \ldots, (2S + 1)N$), counted from right to left:

$$\tilde{a}_{(2S+1)N}(t) < \cdots < \tilde{a}_2(t) < \tilde{a}_1(t).$$

(If a block is on an interval $[i_0, i_1]$ then its position is defined to be $i_1 + 1$; we have followed the notation in [6], section 6.) Let $\bar{Q}^t_{(2S+1)N-i+1}$ denote the size of the ith block ($i = 1, 2, \ldots, (2S + 1)N$); hence, $\bar{Q}_i^t$ is the size of the ith block *if counted from left to right*. Let $\bar{E}_i^t$ be the size of the block of consecutive 0's between the block of 1's corresponding to $\bar{Q}_i^t$ and that corresponding to $\bar{Q}_{i+1}^t$. Then, the statement above is that $\bar{Q}^t_{SN+m}$ and $\bar{E}^t_{SN+m}$ for $m = 1, 2, \ldots, N$ coincide with Q_m^t and E_m^t respectively, when $0 \leqslant t \leqslant CS$, where C is some constant that does not depend on S. Letting S tend to infinity we have

$$Q_m^t = \lim_{S\to\infty} \bar{Q}^t_{SN+m}, \qquad E_m^t = \lim_{S\to\infty} \bar{E}^t_{SN+m} \tag{16}$$

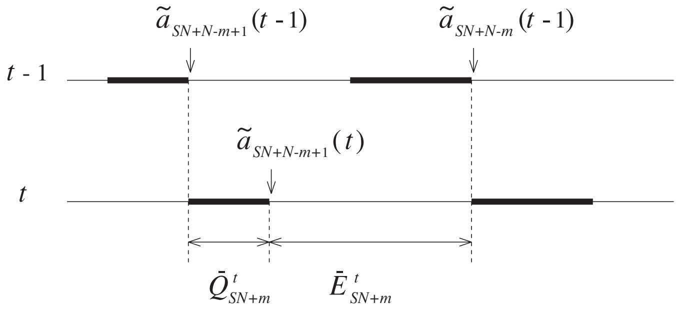

Figure 5. Schematic illustration for (17).

for $m = 1, 2, \ldots, N$ and for all $t \geqslant 0$. It follows that

$$\begin{aligned}\bar{Q}^t_{SN+m} &= \tilde{a}_{SN+N-m+1}(t) - \tilde{a}_{SN+N-m+1}(t-1), \\ \bar{E}^t_{SN+m} &= \tilde{a}_{SN+N-m}(t-1) - \tilde{a}_{SN+N-m+1}(t)\end{aligned} \tag{17}$$

(figure 5) and the right-hand sides of these equations can be written in terms of the initial data, as was done in [6]. Combining (16) and (17) we are able to obtain the desired expression for the solution to the IVP of the udp Toda equation.

3.2. *Proof of theorem 2.1*

Let $\widetilde{\Psi}^t_i$ denote the Psi function of [6] (equation (3) therein) corresponding to our BBS state f_S; therefore $\widetilde{\Psi}^t_i$ depends on S. We have

$$\tilde{a}_i(t) = \widetilde{\Psi}^t_i - \widetilde{\Psi}^t_{i-1}, \tag{18}$$

((7) and (21) in [6]) and an expression for $\widetilde{\Psi}^t_i$ in terms of the initial data (from (33) in [6]):

$$\widetilde{\Psi}^t_{\tilde{m}} = \max_{\substack{J\subset[(2S+1)N] \\ |J|=\tilde{m}}} \left[\sum_{i\in J}(\tilde{\theta}_i + t\widetilde{W}_i) - \sum_{\substack{i,j\in J \\ i\neq j}} \min\{\widetilde{W}_i, \widetilde{W}_j\} \right]$$

for $\tilde{m} = 1, 2, \ldots, (2S+1)N$, where $[k]$ denotes the set $\{1, 2, \ldots, k\}$, $|A|$ the number of elements in A,

$$\tilde{\theta}_i := \tilde{a}_i + 2 \sum_{1\leqslant j\leqslant i-1} \min\{\widetilde{W}_i, \widetilde{W}_j\},$$

$\tilde{a}_i := \tilde{a}_i(0)$ and where $\widetilde{W}_i$ is a positive integer (which in [6] was called the amplitude of the ith block of f_S), defined by $\widetilde{W}_i = \mathbf{W}_{(2S+1)N-i+1}$. Here $\mathbf{W}_i$ is determined in a similar manner as for W_i in the previous section, i.e., from a lattice path corresponding to

$$+\infty, \underbrace{\underbrace{q_1, e_1, \ldots, q_N, e_N}, \ldots, \underbrace{q_1, e_1, \ldots, q_N, e_N}}_{2S\text{ times}}, q_1, e_1, \ldots, q_N, +\infty,$$

which is represented by the graph of a function h defined by

$$h(x) = \begin{cases} x & (x < 0) \\ -x & (0 \leqslant x < q_1) \\ x - 2q_1 & (q_1 \leqslant x < q_1 + e_1) \\ -x + 2e_1 & (q_1 + e_1 \leqslant x < q_1 + e_1 + q_2) \\ \ldots . \end{cases}$$

Let $\mathbf{v}_i$ be the x-coordinate of the ith minimum, and

$$\mathbf{l}_i = \begin{pmatrix}\text{the } x\text{-coordinate of the position where the height,}\\ \text{relative to that at } \mathbf{v}_i\text{, becomes } -1 \text{ for the first time,}\\ \text{when we go along the path leftward starting from } \mathbf{v}_i\end{pmatrix},$$

$$\mathbf{r}_i = \begin{pmatrix}\text{the } x\text{-coordinate of the position where the height,}\\ \text{relative to that at } \mathbf{v}_i\text{, becomes 0 for the first time,}\\ \text{when we go along the path rightward starting from } \mathbf{v}_i;\\ \text{or } +\infty \text{ if there is no such position}\end{pmatrix}.$$

Let $\mathbf{W}_i^L$ denote the maximum height, relative to that at $\mathbf{v}_i$, in the lattice path on the interval $(\mathbf{l}_i, \mathbf{v}_i]$, and let $\mathbf{W}_i^R$ denote the maximum height, relative to that at $\mathbf{v}_i$, in the lattice path on the interval $[\mathbf{v}_i, \mathbf{r}_i)$; or $\mathbf{W}_i^R = +\infty$ if $\mathbf{r}_i = +\infty$. Then $\mathbf{W}_i := \min\{\mathbf{W}_i^L, \mathbf{W}_i^R\}$. $\mathbf{W}_1$ corresponds to the leftmost minimum, and $\mathbf{W}_{(2S+1)N}$ to the rightmost one. (cf Appendix A.2 where another method for computing $\mathbf{W}_i$ is presented.) It follows that $\tilde{a}_{i+N} - \tilde{a}_i = -L$, and, thanks to the assumption (10), that $\widetilde{W}_i = \widetilde{W}_{i+N}$ for $i = 1, \dots, 2SN$. Therefore, we have

$$\begin{aligned}\widetilde{Z}_i := \tilde{\theta}_{i+N} - \tilde{\theta}_i &= -L + 2\sum_{i\leqslant j\leqslant i-N+1} \min\{\widetilde{W}_i, \widetilde{W}_j\}\\ &= -L + 2\sum_{1\leqslant j\leqslant N} \min\{\widetilde{W}_i, \widetilde{W}_j\},\end{aligned}$$

and $\widetilde{Z}_{i+N} = \widetilde{Z}_i$. By virtue of inequalities

$$\sum_{i\leqslant j\leqslant i-N+1} \min\{\widetilde{W}_i, \widetilde{W}_j\} \leqslant \sum_{i\leqslant j\leqslant i-N+1} \widetilde{W}_j = M \left(= \sum_m q_m\right)$$

and the assumption $M < L/2$, we have $\tilde{\theta}_i > \tilde{\theta}_{i+N}$.

Decompose $[(2S+1)N]$ as $[(2S+1)N] = \bigcup T_k$, where $T_k := \{k, N+k, 2N+k, \dots, 2SN+k\}$, $k = 1, \dots, N$. Let $\mathcal{T}_{n_1,\dots,n_N} = \{J \subset [(2S+1)N] \,|\, |J \cap T_k| = n_k (k = 1, \dots, N)\}$ for each $(n_j) = (n_1, \dots, n_N) \in [0, 2S+1]^N$. Then

$$\widetilde{\Psi}^t_{\tilde{m}} = \max_{\substack{J\subset[(2S+1)N]\\ |J|=\tilde{m}}} [\dots] = \max_{\substack{(n_j)\in[0,2S+1]^N\\ \sum n_i=\tilde{m}}} \Big[\max_{J\in\mathcal{T}_{n_1,\dots,n_N}} [\dots]\Big].$$

Since $\tilde{\theta}_i > \tilde{\theta}_{i+N}$ and $\widetilde{W}_i = \widetilde{W}_{i+N}$, the inner bracket $[\dots]$ on the right-hand side attains its maximum at

$$J = J_{n_1,\dots,n_N} := \bigcup_{1\leqslant k\leqslant N} \{jN + k | 0 \leqslant j \leqslant n_k - 1\};$$

hence,

$$\begin{aligned}\widetilde{\Psi}^t_{\tilde{m}} &= \max_{\substack{(n_j)\in[0,2S+1]^N\\ \sum n_j=\tilde{m}}} \left[\sum_{i\in J_{n_1,\dots,n_N}} (\tilde{\theta}_i + t\widetilde{W}_i) - \sum_{\substack{i,j\in J_{n_1,\dots,n_N}\\ i\neq j}} \min\{\widetilde{W}_i, \widetilde{W}_j\}\right]\\ &= \max_{\substack{(n_j)\in[0,2S+1]^N\\ \sum n_j=\tilde{m}}} \left[\sum_{i=1}^N n_i \left(\tilde{\theta}_i + \frac{\widetilde{Z}_i}{2} + \widetilde{W}_i + t\widetilde{W}_i\right) - \sum_{i=1}^N n_i^2 \frac{\widetilde{Z}_i}{2} - \sum_{i=1}^N \sum_{j=1}^N n_i n_j \min\{\widetilde{W}_i, \widetilde{W}_j\}\right],\end{aligned}$$

where in the last line we have used

$$\sum_{i\in J_{n_1,\ldots,n_N}} \tilde{\theta}_i = \sum_{k=1}^{N}\sum_{j=0}^{n_k-1} \tilde{\theta}_{jN+k} = \sum_{k=1}^{N}\left(n_k\tilde{\theta}_k - \frac{n_k(n_k-1)}{2}\widetilde{Z}_i\right)$$

and

$$\sum_{\substack{i,j\in J_{n_1,\ldots,n_N}\\ i\neq j}} \min\{\widetilde{W}_i, \widetilde{W}_j\} = \sum_{i=1}^{N}\sum_{j=1}^{N} n_i n_j \min\{\widetilde{W}_i, \widetilde{W}_j\} - \sum_{i=1}^{N} n_i \widetilde{W}_i.$$

Substituting $SN + N - m$ for $\tilde{m}$ and $S + 1 - n_{N+1-j}$ for n_j and writing

$$a_i := \tilde{a}_{N+1-i}, \qquad W_i := \widetilde{W}_{N+1-i}$$

yields

$$\widetilde{\Psi}^t_{SN+N-m} = \Psi^t_{m,S} + C_S + (S+1)M\cdot t + \left(S+\frac{1}{2}\right)L\cdot m, \tag{19}$$

where

$$\Psi^t_{m,S} := \max_{\substack{(n_i)\in[-S,S+1]^N\\ \sum n_i = m}} \left[\sum_{i=1}^{N} n_i(b_i - tW_i) - \sum_{i=1}^{N}\sum_{j=1}^{N} n_i \Xi_{ij} n_j\right],$$

with b_i and Ξ_{ij} as defined in (12) and (13). C_S is a constant which does not depend on t nor on m. Combining (16), (17), (18) and (19), we obtain

$$Q^t_m = \lim_{S\to\infty}\left(\Psi^t_{m-1,S} - \Psi^t_{m,S} - \Psi^{t-1}_{m-1,S} + \Psi^{t-1}_{m,S}\right),$$
$$E^t_m = \lim_{S\to\infty}\left(\Psi^{t-1}_{m,S} - \Psi^{t-1}_{m+1,S} - \Psi^t_{m-1,S} + \Psi^t_{m,S}\right).$$

The limit $\lim_{S\to\infty}\Psi^t_{m,S}$ exists because the matrix (Ξ_{ij}) is positive definite ([6] section 6). Thus we have arrived at the expression in the theorem.

4. Concluding remark

We remark briefly on the relation between the positive definite matrix $\Xi = (\Xi_{i,j})$ and the ultradiscrete period matrix B which appears in the method of ultradiscretization of the discrete periodic Toda equation [5]. Although both determine the fundamental cycles of the same PBBS, they are substantially different from each other, as is immediately clear from equations (20) and (21).

Let σ be a permutation such that $W_{\sigma(1)} \leqslant W_{\sigma(2)} \leqslant \cdots \leqslant W_{\sigma(N)}$. Write $V_i = W_{\sigma(i)}$. The $N\times N$ matrix $\Xi = \left(\Xi_{i,j}\right)$ has eigenvalues λ_i $(i = 1, 2, \ldots, N)$ where

$$\lambda_1 = \frac{L}{2}, \qquad \lambda_i = \frac{L}{2} - \sum_{1\leqslant k<i-1} V_k - (N-i+2)V_{i-1} \qquad (i = 2, 3, \ldots, N),$$

for the two matrices Ξ and $(\Xi_{\sigma(i),\sigma(j)})$ have common eigenvalues. As the latter has a simple form,

$$\Xi_{\sigma(i),\sigma(j)} = \begin{cases} V_{\min\{i,j\}} & (i\neq j)\\ \frac{L}{2} - \sum_{1\leqslant k<i} V_k - (N-i)V_i & (i = j),\end{cases}$$

the eigenvalues are obtained immediately as above. The corresponding eigenvectors (though we do not need them) of $(\Xi_{\sigma(i),\sigma(j)})$ are

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -N+1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ -N+2 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \ldots, \qquad v_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ 1 \end{pmatrix}.$$

The ultradiscrete period matrix B has the following form ([5], (4.41)):

$$QBQ^T = (-\pi\sqrt{-1}\varepsilon)^{-1} \cdot \mathrm{diag}\left(\frac{2}{1}\lambda_N, \frac{3}{2}\lambda_{N-1}, \ldots, \frac{N+1}{N}\lambda_1\right), \tag{20}$$

where

$$Q = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & -1/N & 1 \end{pmatrix} \times \cdots \times \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & -1/3 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1/3 & 0 & \ldots & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1/2 & 1 & 0 & \ldots & 0 \\ -1/2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1/2 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

Hence, we obtain the relation

$$\det((-\pi\sqrt{-1}\varepsilon)B) = (N+1)\det \Xi. \tag{21}$$

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Appendix. Another method for computing W_i and $\mathbf{W}_i$

In this appendix, another method for computing W_i (for an initial state of the udp Toda, or the PBBS) and $\mathbf{W}_i$ (for that of the ultradiscrete Toda molecule, or the BBS) is presented. This method was in fact adopted as the definition of $\mathbf{W}_i$ in our previous paper [6].

A.1. W_i: the periodic case

Let $(q_1, e_1, \ldots, q_N, e_N)$ be a $2N$-tuple of positive integers. Let f be a finite sequence which consists of q_1 1's followed by e_1 0's, followed by q_2 1's, and so on. In what follows we always take the periodic boundary condition into account when thinking about sequences of length $L = \sum q_i + \sum e_i$, so that, for example, in $f = (f(1), f(2), \ldots, f(L))$ the position L is supposed to be adjacent to position 1.

1. Obtain $\tilde{f}^{(k)}$'s from the following algorithm:

`Input`: f, a sequence of 1's and 0's of length L with $\sharp f^{-1}(1) < \sharp f^{-1}(0)$.
`Output`: a finite sequence $\tilde{f}^{(0)}, \tilde{f}^{(1)}, \ldots$ of sequences of 1's, 0's and ' "s (SPACEs) of length L.
`Begin`

Set $\tilde{f}^{(0)} = f$ and $k = 0$.

`While` $\tilde{f}^{(k)}$ still contains 1's `do`

- Set $g = \tilde{f}^{(k)}$.
- In each consecutive sequence of 1's in g where SPACEs between 1's are to be skipped if they exist, change the leftmost 1 to a SPACE. In each consecutive sequence of 0's in g where SPACEs between 0's are to be skipped if they exist, change the rightmost 0 to a SPACE. Update g.
- Set $\tilde{f}^{(k+1)} = g$. Increment the value of k by 1.

`End`

If the `while` loop is repeated K times, then we have $\tilde{f}^{(0)}, \tilde{f}^{(1)}, \ldots, \tilde{f}^{(K-1)}$.

2. Let $a_1, a_2, \ldots, a_N$ denote the elements of the set $\{n | f(n-1) = 1, f(n) = 0, n = 1, 2, \ldots, L\}$ where $a_1 < a_2 < \cdots < a_N$. For each k ($k = 0, 1, \ldots, K-1$) and i ($i = 1, 2, \ldots, N$), insert '|' (called the i^{th} '|') into the position between the digits (and/or SPACE(s)) of $\tilde{f}^{(k)}$ at $a_i - 1$ and a_i.
3. For each i ($i = 1, 2, \ldots, N$), let L_i be the smallest number k such that the three digits of $\tilde{f}^{(k)}$ to the left of the ith '|' (skipping SPACEs if there are) are 001; if there is no such number k we set $L_i = 0$.
4. For each i ($i = 1, 2, \ldots, N$), let R_i be the smallest number k such that the two digits of $\tilde{f}^{(k)}$ to the right of the ith '|' (skipping SPACEs if there are) are 01; if there is no such number k we set $R_i = \infty$.
5. Then, $W_i = \min\{L_i + 1, R_i + 1\}$ ($i = 1, 2, \ldots, N$).

For example, if $N = 4$ and an 8-tuple $(1, 4, 3, 2, 4, 6, 4, 7)$ is given, then

$$f = 100001110011110000001111 0000000$$

and we have a table:

```
            1st       2nd     3rd        4th
f̃(0) =  1|0000111|001111|0000001111|0000000
f̃(1) =   |000    11|0    111|00000   111|000000
f̃(2) =   |00      1|       11|0000     11|000000
f̃(3) =   |0        |       11|000       1|000000
f̃(4) =   |         |        1|00         |000000
f̃(5) =   |         |         |00         |00000.
```

Hence, $L_1 = 0$, $R_1 = 3$, $L_2 = 2$, $R_2 = 1$, $L_3 = 4$, $R_3 = \infty$, $L_4 = 3$ and $R_4 = \infty$, and hence $W_1 = 1$, $W_2 = 2$, $W_3 = 5$ and $W_4 = 4$. The W_i's coincide with those obtained by (8) in section 2.

A.2. $\mathbf{W}_i$*: the nonperiodic case*

Let $(q_1, e_1, \ldots, q_{N-1}, e_{N-1}, q_N)$ be a $(2N-1)$-tuple of positive integers. Let f be an infinite sequence which consists of an infinite number of 0's, followed by q_1 1's, followed by e_1 0's, followed by q_2 1's, ..., followed by q_N 1's, followed by an infinite number of 0's, where the leftmost one is assumed to be at position 1 (i.e., if f is considered as a mapping of $\mathbb{Z}$ into $\{0, 1\}$, then $f(1) = 1$ and $f(i) = 0$ for $i \leqslant 0$).

1. Obtain $\tilde{f}^{(k)}$'s from the following algorithm:

`Input`: f, an infinite sequence of 1's and 0's with $\sharp f^{-1}(1) < \infty$.
`Output`: a finite sequence $\tilde{f}^{(0)}, \tilde{f}^{(1)}, \ldots$ of infinite sequences of 1's, 0's and ' 's (SPACEs).
`Begin`

Set $\tilde{f}^{(0)} = f$ and $k = 0$.

`While` $\tilde{f}^{(k)}$ still contains 1's `do`

- Set $g = \tilde{f}^{(k)}$.
- In each consecutive sequence of 1's in g where SPACEs between 1's are to be skipped if they exist, change the leftmost 1 to a SPACE. In each consecutive sequence of 0's of *finite size* in g where SPACEs between 0's are to be skipped if they exist, change the rightmost 0 to a SPACE. Update g.
- Set $\tilde{f}^{(k+1)} = g$. Increment the value of k by 1.

`End`

If the `while` loop is repeated K times, then we have $\tilde{f}^{(0)}, \tilde{f}^{(1)}, \ldots, \tilde{f}^{(K-1)}$.

2. Let $a_1, a_2, \ldots, a_N$ denote the elements of the set $\{n | f(n-1) = 1, f(n) = 0, n \in \mathbb{Z}\}$ where $a_1 < a_2 < \cdots < a_N$. For each k $(k = 0, 1, \ldots, K-1)$ and i $(i = 1, 2, \ldots, N)$, insert '|' (called the ith '|') into the position between the digits (and/or SPACE(s)) of $\tilde{f}^{(k)}$ at $a_i - 1$ and a_i.
3. For each i $(i = 1, 2, \ldots, N)$, let L_i be the smallest number k such that the three digits of $\tilde{f}^{(k)}$ to the left of the ith '|' (skipping SPACEs if there are) are 001; if there is no such number k we set $L_i = 0$.
4. For each i $(i = 1, 2, \ldots, N)$, let R_i be the smallest number k such that the two digits of $\tilde{f}^{(k)}$ to the right of the ith '|' (skipping SPACEs if there are) are 01; if there is no such number k we set $R_i = \infty$.
5. Then, $\mathbf{W}_i = \min\{L_i + 1, R_i + 1\}$ $(i = 1, 2, \ldots, N)$.

These $\mathbf{W}_i$'s coincide with those in section 3.2 (with N being replaced by $(2S+1)N$). This $\mathbf{W}_i$ was written as W_{N-i+1} in the previous paper [6].

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